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An Applied Mathematician's Apology

Bruce Henry[1]
[1] School of Mathematics and Statistics, UNSW, Sydney, Australia

E-mail: b.henry@unsw.edu.au

Abstract
In this article I provide a personal perspective on what Applied Mathematics is and why it is important. The academic discipline of Applied Mathematics sits somewhere between, and across, the academic discipline of Pure Mathematics and the pragmatism of applications. Much of the domain of Applied Mathematics is abstract and may not appear to be useful for real world applications. However it is through such abstractions that new mathematics is created and the rich mapping between the physical universe and mathematics, which is necessary for applications, is advanced.

Introduction
Let me begin by conveying my sincere congratulations to Professor David Wilson for his very nice work in applying mathematics, which has been recognized by the Royal Society of New South Wales with the Edgeworth David Medal for 2013.

In his invited discourse in this issue David provides a short critique of the academic discipline of Applied Mathematics. In his view this discipline has diverged from the pursuit of applying mathematics to real world systems; becoming instead an activity of mathematical exploration that "does not advance, or provide insight into, the application and nor does it advance fundamental mathematical theory." David goes further by contrasting this with the academic discipline of Pure Mathematics. Here he says, there is "no deception about the uselessness of their complex theorems and mathematical research." David urges mathematical scientists, with interests in applications of mathematics, to reduce complexity in their analysis and to concentrate on providing clear and timely advice. In illustration of his message he draws attention to Daniel Bernoulli's seminal paper on mathematical epidemiology, a translation of which can be found in Blower (2004).

I welcome the opportunity to provide a defence of the academic discipline of Applied Mathematics.

An Apology[1]
I do not fully agree with David's description of the academic discipline of Applied Mathematics, but his observations are not entirely without justification. There is a vast literature of Applied Mathematics that may be considered permutations, involutions and explorations of complex model equations that are often far removed from any real world applications. There is also a significant literature that is more sharply focused on applications. It is my strong view that the academic pursuit of Applied Mathematics, including mathematical explorations, is vitally important. This academic pursuit enables the more practical pursuit of applying mathematics. One might intercede here and say that it is Pure

[1] The title is a variant of Godfrey Harold Hardy's 1940 essay "A Mathematician's Apology". Readers of Hardy's essay will observe that while he strongly defended Pure Mathematics he made rather disparaging remarks about Applied Mathematics.

Mathematics that enables the practical pursuit of applying mathematics. There is truth in this too; the kingdom needs a King and a Queen.

In order to present a defence of the academic discipline of Applied Mathematics it is useful to attempt to define what it is that Applied Mathematicians do[2]. There is no absolute definition that I am aware of, and in some ways a definition might be regarded as a philosophical position. As context, for my definition, I consider an expanding universe of mathematics, created purely from imagination[3], that exists in parallel with the physical universe that we inhabit and create. Pure Mathematicians explore and extend the universe of mathematics, developing and imagining new vistas. Applied Mathematicians build on and further develop the universe of mathematics to enable mathematics to be used to provide understanding, prediction and improvements, including technological developments, in the physical universe. Pure Mathematicians and Applied Mathematicians are of course both mathematicians and it is not possible to define an absolute boundary between them. Mathematical scientists who are applying mathematics to inform policy makers are dependent on structures created by both Pure Mathematicians and Applied Mathematicians. The number of mathematical scientists has grown enormously over the past two centuries extending through the quantitative disciplines of economics, computer science, engineering, actuarial science, bioinformatics etc. The number of Applied Mathematicians and Pure Mathematicians is small compared with the cohort of mathematical scientists applying mathematics.

As an academic Applied Mathematician, I have never used my knowledge of mathematics to attempt to influence decision-making and hence my comments on this will be based on logic rather than experience. It certainly makes sense to explain things simply, with simple analysis, if you are attempting to inform policy makers who are not very mathematically literate. However this does not mean that the analysis itself should be simple. With time constraints it makes sense to reduce explorations but it is also makes sense to be mindful of the well known aphorism, often attributed to Albert Einstein, "Everything should be made as simple as possible, but not simpler"[4]. There can be more than one model, and one set of analysis; one, which is a simplification to communicate the essence to policy makers, and another, which has sufficient mathematical complexity to convince mathematical scientists of its validity. The schematic figure eight flight trajectories for the Apollo moon missions come to mind in this context, contrasted with the complicated mathematical calculation of the actual Earth-Moon-Earth trajectory of the spacecraft. The figure eight

[2] Hardy posed the question "How do pure and applied mathematicians differ from one another?" and he immediately followed this with "This is a question which can be answered definitely and about which there is general agreement among mathematicians." However he didn't provide a direct answer to the question.

[3] There is no consensus on whether mathematics are created or discovered. I have adopted a non-Platonist philosophical position on this.

[4] When Einstein developed a mathematical model for general relativity he found it necessary to go beyond the mathematics of vector analysis and Euclidean geometry, and to use the mathematics of tensor analysis. Tensor analysis was the mathematics that was needed to be able to describe the geometry of a four-dimensional space-time. Tensor analysis would never have been an option without the mathematical explorations that developed it.

trajectory is a cartoon. The moon is in motion relative to the position of the Earth and it is travelling about five times faster than the spacecraft when the spacecraft gets near the moon (Crenshaw, 2010).

It is not difficult to find examples where simple analysis fails. If you plot Olympic Gold Medal winning and Olympic Silver Medal winning 100-metre sprint times as a function of year, since 1896 you will find an approximate linear fit between Gold Medal winning times and the year, and a different linear fit between Silver Medal winning times and the year. In each case the medal winning times decrease approximately linearly as a function of time, with the slope of the best fit for Silver Medal times being greater than the slope of the best fit for Gold Medal times. Having different slopes means that the lines must cross over at some future point in time. Naïve advice based on this simple analysis would then suggest that at some point in the future, Silver Medal athletes would be running faster than Gold Medal athletes in this event. This conclusion is of course fanciful nonsense but it is not entirely a straw man argument. A similar simple analysis, published in Nature (Tatem et al, 2004), led to the conclusion that women would have faster sprint times than men in the 100-metres event in the 2156 Olympics.

Those mathematical scientists involved in applying mathematics to advise policy makers should be prepared to embrace the most relevant mathematics in their modelling. They should be prepared to engage in cutting edge mathematics, if needed, and they should be prepared to have dialogues with Applied Mathematicians and Pure Mathematicians. Daniel Bernoulli's work on epidemiology followed this paradigm. The aim of Bernoulli's epidemiology paper (Blower, 2004), published in 1766, was to find the increase in life expectancy of a newborn, if there was inoculation against smallpox. His analysis might be considered simple for some practicing mathematicians today, but it was not simple at the time. It was disputed by a contemporary of Bernoulli, the Applied Mathematician, D'Alembert (Dietz and Heesterbeck, 2002). Daniel Bernoulli was by no means an ordinary mathematician. He also had access to state of the art mathematical methods through contact with his uncle Jakob, and earlier, with his father Johann. Daniel Bernoulli's paper contained the first formulation of a mathematical model for the spread of an epidemic in terms of ordinary differential equations. After some elegant analysis, Bernoulli reduced the mathematical model to a special type of differential equation that he could then solve using a method devised by his uncle, Jakob Bernoulli. This special type of differential equation belongs to a class of differential equations now known as Bernoulli equations. The development of methods of solution for that class of differential equations is a classic example of exploration in Applied Mathematics.

The techniques for simple mathematical analysis do not spontaneously come into existence. They follow earlier mathematical explorations. Sometime ago I used simple analysis to calculate the possibility of executing a fundamental surfing manoeuvre; dropping down the face of a wave, doing a bottom turn, and returning to the top of the wave before it breaks (Henry and Watt, 1998). A slightly simpler problem of this type (the *Brachistochrone Problem*) was originally set as a challenge "to the most clever mathematicians in the world" by Johann Bernoulli in 1696. Solutions were obtained by the giants in mathematics; Isaac

Newton, Gottfried Leibniz, Guillaume de L'Hopital, Jacob Bernoulli and Johann Bernoulli. At the time their solutions might have appeared as complex explorations far removed from any significant real physical system – the shortest time path for a point like object to slide from rest without friction under the action of a constant force. However their Applied Mathematics explorations led to the formulation of the calculus of variations, and this is now a mainstay of all optimization problems where the object is to find a function that maximizes or minimizes some specified constraints.

As a general principle it makes sense to reduce the complexity of the analysis if possible, especially if timely advice is important. But it is also important to provide accurate advice, or at least to provide advice on the level of accuracy. David alludes to this but does not give it prominence; "even if the analyses are only around 70% complete or precise but are communicated clearly and through appropriate channels then they are likely to inform the decision-making". This 70% idea is like the Pareto 80/20 principle for time management, which roughly states that for many situations 80% of the complete or precise result may be obtained from 20% of the effort needed to get the complete or precise result. This can guide time allocation but in providing advice the accuracy of the advice should play a prominent role. The advice should contain reliability estimates. This may necessitate going beyond simple analysis.

The Global Financial Crisis of 2007, 2008, brought the importance of accurate advice into sharp focus. It is generally accepted that a major contributor to this crisis was inaccurate advice from financial advisors who did not properly understand risk (Taleb & Martin, 2012). Some of this inaccurate advice was based on a formula that was derived from the simplifying assumption that the price of Credit Default Swaps was correlated with the price of mortgage-backed securities (Salmon, 2009). The formula was popular because it could deliver quick and decisive advice but the simplifying assumption was flawed.

It may be possible to go beyond simple analysis and still provide clear and timely advice that is far more reliable than simple analysis could provide. This is the case in modern weather forecasting. An example of simple analysis in this context is the persistence model for weather forecasting. This model predicts that tomorrow's weather will be the same as today's weather. Modern weather forecasting is not simple analysis, but it can be done in a timely fashion to provide clear and accurate advice. It evolved to this level of sophistication after two hundred years of Applied Mathematics explorations in partial differential equations, nonlinear dynamics and computational mathematics.

Applied Mathematics is the mapping, and further development, connecting and extending mathematics with the physical universe, including the creation of new technologies. This definition connects a little with Galileo's view that the universe is like a grand book, written in the language of mathematics. Galileo remarks that "it is humanely impossible to comprehend a single word" "until we have learnt the language and become familiar with the characters in which it is written". In this context, Pure Mathematics could be construed as the language of the universe with Applied Mathematics playing the role of an interpreter and an author, enabling the physical universe to be understood and extended through new technologies. As an example, the discovery or invention of natural numbers could be considered as Pure Mathematics. The discovery and development of natural numbers for the purpose of counting and ordering could be considered as Applied Mathematics. Using natural numbers for counting and ordering could be considered as applying mathematics. I am not suggesting that this is historically how things evolved in this example. It is almost certainly true that counting preceded the creation, or was the creation of natural numbers. The marks on the Ishango bone[5], believed to be 20,000 years old, may be one of the earliest examples of a counting system. Prime numbers, the fundamental theorem of arithmetic, and the prime number theorem are in the domain of Pure Mathematics, but the creation of public key encryption methods based on prime numbers is Applied Mathematics. Applying mathematics to the security of Internet financial transactions is dependent on such methods.

In general, discoveries or inventions in Pure Mathematics lead to discoveries and inventions in Applied Mathematics and vice versa. The fundamental elements of calculus, derivatives and integrals, may be considered as the domain of Pure Mathematics but the discovery and development of calculus to describe rates of change in real world phenomena is Applied Mathematics. Again there is no definitive boundary but this can be used as a guide. In this case we might consider Leibniz's calculus as Pure Mathematics and Newton's calculus as Applied Mathematics. Some historians have argued that much of what we would regard as Pure Mathematics evolved out of Applied Mathematics through the 19th and 20th centuries (Maddy, 2008).

As a further example, the Navier-Stokes partial differential equations, which were developed by Claude-Louis Navier in 1822 and George Gabriel Stokes in 1854 are fundamental to all modern weather forecasting models. Their study, development, and implementation in this context is Applied Mathematics. However the determination of whether or not smooth solutions always exist for these equations in three-dimensions is currently an open problem in Pure Mathematics[6]. Weather forecasts by any of the myriad providers are examples of applying mathematics.

As a final example, let me discuss an area of current interest in Applied Mathematics that I am somewhat familiar with. This is the area of fractional calculus. The history of fractional calculus in Pure Mathematics goes back to the foundations of calculus more than three hundred years ago. One of the founders of Calculus, Gottfried Leibniz, in a letter to Guillaume de L'Hôpital in 1695 posed the question (Miller and Ross, 1993): *"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?"* At the time, L'Hôpital was writing the first textbook on calculus[7]. Many Pure Mathematicians became interested in this problem and developed expressions for fractional derivatives and fractional integrals. The fractional calculus first appeared in Applied Mathematics when

[5] http://en.wikipedia.org/wiki/Ishango_bone

[6] This is one the seven Millennium Prize Problems established by the Clay Mathematics Institute in 2000 as an important classic question whose solution deserves a million dollar prize.

[7] *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* (*Analysis of the infinitely small to understand curves*) (1696).

Niels Abel made mention of it in his 1823 paper on the *Tautochrone Problem*. This problem was to find the curve for which the time taken for a point like object to slide from rest, under the influence of gravity, and without friction, to reach the lowest point is independent of the starting point. The optimal curve is a cycloid, which is also the solution to the *Brachistochrone Problem*.

It is only in recent decades that fractional calculus has started to have a significant impact in Applied Mathematics. The motivation for this interest has its origins in many physical and biological experiments that reported diffusion of particles some orders of magnitude slower than that anticipated by Albert Einstein's *Theory of Brownian Motion*. A reconsideration of diffusion, derived from the mathematics of continuous time random walks, and taking into account the effects of particle trapping and obstacles, has led to the creation of new mathematical models of diffusion, including new diffusion equations with fractional order temporal derivatives. The new models, which can provide a better fit to data, have stimulated a lot of explorations in Applied Mathematics. These explorations are now starting to flow back to Pure Mathematics.

Most mathematical biologists seeking to apply mathematics to problems of diffusion are not yet equipped to venture far beyond Einstein's model of Brownian motion. It may be sometime yet before the fractional calculus enters the domain of what any applied practitioner might regard as simple analysis. However, without the pursuits of Pure Mathematics and Applied Mathematics it never could. There is currently a lot of exploration in Applied Mathematics developing fractional calculus models with only tenuous links back to any real world system. Some of this, on its own, indeed most of it, may well turn out to be useless. But it is this overall level of activity that produces a breeding ground that is necessary for creating new, and potentially useful, results. This activity also provides a platform to train the next generation of scientists who will be able to incorporate newly created methods into possible simple analysis in a multidisciplinary setting.

Related to the example above, I would like to mention in passing that it is generally the case among tertiary education providers around the world to provide less mathematics training for students in the biological sciences than those in the physical sciences and engineering. This should not be the case. We should not limit those applying mathematics in the biological sciences to a handful of tools enabling simple analysis.

We live in challenging times. The human population of more than seven billion is altering its global environment and climate; the well-being of our economic and financial systems is largely predicated on future growth that is unsustainable; our global connectedness through airline networks, and internet networks, make us vulnerable to global shocks; the demands on our medical systems and transport systems is exceeding capacity; the emergence of terrorism on a global scale poses enormous threats to security. Meeting these challenges will require intellectual advances from science, medicine, engineering, finance and business.

One of the global tasks of mathematicians is to extend the universe of mathematics and to provide the mathematical training that will help to underpin and enable these challenges to be met. Mathematics is not static. It is not something that we have and know in its entirety. It is something that is evolving. It is created by Pure Mathematicians and by Applied Mathematicians through explorations. The activities in our world are increasingly being underpinned by mathematics, and decision makers are increasingly reliant on advice that is underpinned by mathematics. There is no doubt that clear and timely advice, based on accurate and uncomplicated mathematical analysis, will be valuable and sought after. But so will the fruits of mathematical explorations, offering methods of analysis and prediction not yet imagined.

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References

Blower, S. (2004) An attempt at a new analysis of the mortality caused by smallpox and of the advantages on inoculation to prevent it, *Reviews in Medical Virology*, 14, 275-288.

Crenshaw, J. (2010) Fly me to the moon; *Embedded,* http://www.embedded.com/electronics-blogs/programmer-s-toolbox/4211093/Fly-me-to-the-Moon

Dietz, K. and Heesterbeek, J. A. P. (2002) Daniel Bernoulli's epidemiological model revisited, *Mathematical Biosciences*, 180, 1-21.

Hardy, G. H. (1940) A mathematician's apology; http://www.math.ualberta.ca/mss/

Henry, B. I. and Watt, S. D. (1998) Surfing Brachistochrones; *Parabola,* 34, Issue 3.

Maddy, P. (2008) How applied mathematician's became pure, The Review of Symbolic Logic, 1, 16-41.

Miller, K. S. and Ross, B. (1993) *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons. Inc., New York.

Salmon, F. (2009) Recipe for Disaster: The Formula That Killed Wall Street; *Wired Magazine*, Issue 17-03, 2009.

Taleb, N. and Martin, G. (2012) How to Prevent Other Financial Crises; *SAIS Review*, XXXII, No 1.

Tatem A. J., Guerra C. A., Atkinson P. M. and Hay S. I. (2004) Athletics: Momentous sprint at the 2156 Olympics? *Nature*, 431. 525.

Bruce Henry is Head of the School of Mathematics and Statistics at the University of New South Wales. He is an Applied Mathematician with more than seventy international journal publications across many different discipline areas, including: applied mathematics, biology, condensed matter physics, geology, materials science, mathematical physics, plasma physics, stochastic process, neuroscience and vision science. For the past fifteen years most of his research has been focussed on developing and combining continuous time random walks, fractional calculus and stochastic analysis, to provide model equations for the motions of particles that react and diffuse in environments with traps, obstacles and external forces.

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