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On an f(R) Theory of Gravity

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Abstract: We attempted to develop a higher-order theory of gravitation based on a Lagrangian density consisting of a polynomial of scalar curvature, R to obtain gravitational wave equations conformally flat. In this theory, it is desirable to study the gravitational field of a spherically symmetric mass distribution and the motion of particle to bring out the effect of modification of general relativity. In the context it is found that the spherically symmetric metric is not asymptotically flat as r tends to infinity and, in case of orbital motion of the planet, it turns out that it differs from Einstein case by having an additional term, though of small magnitude, in the equation. This term does not contribute to produce observable effect as such the precession of the perihelion is consistent with observation.

Keywords: General Relativity

INTRODUCTION

We investigated an f(R) theory of gravity in the context of general relativity. However, this theory in the framework of Palatini approach as solution to the problem of the observed accelerated expansion of the universe is discussed in Capozziello et al. by considering two physically motivated popular choices for f(R), that is, power law, $f(R) = \beta R^N$ and logarithmic, $f(R) = \alpha \log R$. This give rise to cosmological models comprising only standard matter and undergoing a present phase of accelerated expansion but the deceleration parameter is higher than what is measured in the concordance ΛCDM model. The ΛCDM model is also plagued by many problems on different scales. If interpreted as vacuum energy, Λ is up to 120 orders of magnitude smaller than the predicted value.

In this framework, there is also the attractive possibility to consider the Einstein general relativity as a particular case of a more fundamental theory. This is the underlying philosophy of what are referred to as f(R) theories. In this case, Friedmann equations have to be given away in favour of a modified set of cosmological equations that are obtained by varying a generalized gravity Lagrangian where the scalar curvature R has been replaced by a generic function f(R). The usual general relativity is recovered in the limit f(R) = R, while completely different results may be obtained for other choices of f(R). While in the weak field limit the theory should give the usual Newtonian gravity, at cosmological scales there is an almost complete freedom in the choice of f(R). This leaves open the way to a wide range of models.

On the other hand, the non-conformal invariance of gravitational waves which are an inevitable consequence of Einstein theory of gravitation motivated us Pandey 1983, Pandey 1988, Grishchuk 1977 to modify the Einstein theory by choosing f(R) as a polynomial in R of a finite number of terms without associating any other field except gravitation. Therefore, we took the Lagrangian in the form

$$\mathcal{L} = R + \sum_{n=2}^{N} Cn\{(l^2 R)^n / 6l^2\}$$

$$\text{or equivalently} \quad \mathcal{L} = R + \sum_{n=2}^{N} a_n R^n \tag{1}$$

where l is the characteristic length and C_n are the dimensionless coefficients corresponding to n introduced to nullify the manifestation of gravitation. The values of $n = 0$ and 1 result in Hilbert Lagrangian, that is, Einstein theory. Therefore n begins from $n = 2$ onwards.

This choice of f(R) should not be disturbing because it is an observational fact that our universe is not asymptotically flat. There is enough matter on our past light cone to cause it to refocus. The total energy of the universe is exactly zero, the positive energy of gravitation and the matter particle being exactly compensated by the negative gravitational potential energy. That is why the universe is expanding. Also the unitarity is not well defined except in scattering calculations in asymptotically flat spaces.

Therefore, the paper is organised as follows. The field equations of this f(R) theory under consideration are given in section 2. Section 3 deals with an attempt to find the gravitational field surrounding a spherically symmetric mass distribution at rest while the equation of motion of a particle in this field is the subject matter of section 4. In the last section we give concluding remarks on the results of this f(R) theory.

FIELD EQUATIONS

A quite interesting and fascinating scenario predicts that standard matter is the only ingredient of the cosmic pie as it is indeed observed, but the Einsteinian general relativity breaks down at the present small curvature scale. As a result we generalize the action as Pandey 1983, Grishchuk 1977, Pandey 2001

$$A = \int (\mathcal{L}/\kappa + \mathcal{L}_s)\,\mathrm{d}^4x \tag{2}$$

with $\mathcal{L}_s$ standing for the source Lagrangian density to obtain the graviton equations in the background of Friedmann universe having scale factor $a(\eta)$ as:

$$\mu'' + \mu[n^2 - a''/a] = 0 \tag{3}$$

An application of variational principle to this action yields the field equations as:

$$R^{uv} - \frac{g_{uv}R}{2} + \sum_{n=2}^{N} n a_n R^{n-1}\Bigg[R_{uv} - \frac{Rg_{uv}}{2n} - \frac{n(n-1)}{R}(R_{;u;v} - g_{uv}\Box R) - \frac{(n-1)(n-2)}{R^2}(R_{;u}R_{;v} - g_{uv}R_{;\alpha}R^{;\alpha})\Bigg] = \kappa T_{uv} \tag{4}$$

Here $rT_{uv} = \sqrt{(-g)}(\delta\mathcal{L}_s/\delta g^{uv})$ (eqn. 5) stands for the energy-momentum tensor responsible for the production of the gravitational potential g_{uv}. It can easily be seen that $T^v_{u;v} = 0$ (eqn. 6) holds for these field equations as it is in case of Einstein general relativity. Again it should be noted that $1 + na_nR^{n-1} \neq 0$ or equivalently,

$$1 + 2a_2R + 3a_3R^2 + 4a_4R^3 + \dots + Na_nR^{N-1} \neq 0 \tag{7}$$

because of the Cauchy problem. This fact is important in studying the completeness of geodesic in higher-order theory of gravitation.

SPHERICALLY SYMMETRIC FIELD

It is interesting to know in this theory of gravity the gravitational field surrounding a spherically symmetric mass distribution at rest. Obviously the gravitational field would have spherical symmetry. We require the field to be static, that is, it should be both time independent and unchanged by time reversal. So, we consider

$$\mathrm{d}s^2 = e^{N(r)}\,\mathrm{d}t^2 - e^{L(r)}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{8}$$

where functions $N(r)$ and $L(r)$ are to be determined by using the field equations of f(R) gravity. In vacuum where $T_{uv} = 0$, we get

$$R^0_0 - R/2 = R^1_1 - R/2 = R^2_2 - R/2 = R^3_3 - R/2 = \Psi(r) \tag{9}$$

$$\text{where}\quad \Psi(r) = \frac{\sum_{n=2}^{N} a_n(2 - n/2)R^n}{1 + \sum_{n=2}^{N} na_nR^{n-1}} \tag{10}$$

The functions $N(r)$ and $L(r)$ are seen to satisfy from $R^0_0 - R/2 = R^1_1 - R/2$ $N(r) = -L(r)$ (eqn. 11). Again $R^0_0 - R/2 = \Psi(r)$ (eqn. 12). So we find

$$e^{-L} = 1 + \frac{k}{r} + \frac{1}{r}\int r^2\Psi(r)\,\mathrm{d}r \tag{13}$$

where k is a constant which can be determined from the fact that at large distances the g_{00} component of the metric must conform to Newtonian potential, that is $1 - 2\varphi$. If M is the central mass then $k = -2MG$ (eqn. 14). leading to

$$\mathrm{d}s^2 = \left(1 - \frac{2MG}{r} + \frac{1}{r}\int r^2 \Psi(r)\,\mathrm{d}r\right)\mathrm{d}t^2 - \mathrm{d}r^2 \Big/ \left(1 - \frac{2MG}{r} + \frac{1}{r}\int r^2 \Psi(r)\,\mathrm{d}r\right) - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 \quad (15)$$

It is to be noted that M is the total mass of the system. The mass energy contributed by the gravitational field is to be included in M. Clearly, in view of the principle of equivalence, the gravitational mass of the system which produces the field (15) is, in fact, equal to the inertial mass of the system The other equations in (9) are satisfied by equations (11) and (13).

Now we turn our attention towards equation (10). The denominator of equation (10) is non-vanishing due to equation (7) and can, therefore, be expanded in powers of r. Thus

$$\Psi(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + \dots \quad (16)$$

Since the contribution of $\Psi(r)$ is very small, then

$$\frac{1}{r}\int r^2 \Psi(r)\,\mathrm{d}r = \frac{b_0 r^2}{3} + \frac{b_1 r^3}{4} + \frac{b_2 r^4}{5} + \dots \quad (17)$$

$$\text{yields} \quad e^N = e^{-L} \approx 1 - \frac{2MG}{r} + \frac{b_0 r^2}{3} \quad (18)$$

by retaining only the first term in equation (17). Therefore the metric (8) becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2MG}{r} + \frac{b_0 r^2}{3}\right)\mathrm{d}t^2 - \mathrm{d}r^2 \Big/ \left(1 - \frac{2MG}{r} + \frac{b_0 r^2}{3}\right) - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 \quad (19)$$

It can easily be seen that the metric (19) is not asymptotically flat due to the presence of term $b_0 r^2/3$. Again, it can be seen that in this f(R) theory of gravitation the space-time, by virtue of equation (7), will no longer be asymptotically flat at larger distances. Again the appearance of term $b_0 r^2/3$ is worth comparing with the Schwarzschild solution, that is,

$$\mathrm{d}s^2 = \left(1 - \frac{2MG}{r} - \frac{\lambda r^2}{3}\right)\mathrm{d}t^2 - \mathrm{d}r^2 \Big/ \left(1 - \frac{2MG}{r} - \frac{\lambda r^2}{3}\right) - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\phi^2 \quad (19a)$$

of the Einstein field equation with cosmological constant, that is, $R_u^v - (1/2)\delta_u^v R = -\Lambda\delta_u^v$. As such the contribution due to modification is behaving as a cosmological constant which is very small and can be taken to correspond to a cosmological correction to the Newtonian potential. For the motion of planets the cosmological correction is completely insignificant. Further, even if we include the second term or more of equation (17), the qualitative picture, for instance the asymptotic flatness, remains unchanged. Quantitatively however the values of the metric potentials will differ.

EQUATION OF MOTION

The equation of motion of a particle in a gravitational field is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0 \quad (20)$$

and the quantity

$$g_{ik}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = \text{constant} = 1 \quad (21)$$

is a constant of motion. Therefore it can be regarded as a first integral of the equation of motion.

Now we consider the equation of motion of a particle or planet in the gravitational field of (8). They are:

$$\ddot{t} + N'\dot{r}\dot{t} = 0 \quad (22)$$

$$\ddot{r} + \frac{1}{2}N'e^{N-L}\dot{t}^2 + \frac{1}{2}L'\dot{r}^2 - re^{-L}\dot{\theta}^2 - r^2\sin^2\theta e^{-L}\dot{\phi}^2 = 0 \quad (23)$$

$$\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} - (\sin\theta\,\cos\theta)\dot{\phi}^2 = 0 \quad (24)$$

$$\ddot{\theta} + \frac{2\dot{r}\dot{\theta}}{r} + 2(\cot\theta)\dot{\theta}\dot{\phi} = 0 \quad (25)$$

where $\dot{r} = dr/d\tau$ and $N' = dN/dr$.

Now we assume that orbit is in the $\theta = \pi/2$ plane. So, initially $\dot{\theta} = 0$ and equation (24) yields $\ddot{\theta} = 0$ This means that the orbit remains in this plane. Further equations (22) and (25) lead to

$$\dot{\phi} = A/r^2 \quad (26)$$

and

$$\dot{t} = Be^{-N} \quad (27)$$

where A and B are constants. As pointed out earlier, the equation (21) is the first integral, we take it and ignore equation (23). Therefore

$$e^N\dot{t}^2 - e^L\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 = 1 \quad (28)$$

which with $\theta = \pi/2$ and $\dot{\theta} = 0$ gives

$$\frac{B^2}{e^N} - \frac{\dot{r}^2}{e^{-L}} - \frac{A^2}{r^2} = 1 \quad (29)$$

by virtue of equations (26) and (27). Now, changing $r = 1/u$ and making use of equation (15), the equation (29) reduces to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u - \frac{GM}{A^2} - 3GMu^2 - \Phi(u) = 0 \quad (30)$$

where

$$\Phi(u) = \left\{(1 + A^2u^2)\frac{\mathrm{d}}{\mathrm{d}\phi}[u\int\frac{\Phi(u)}{u^4}\,\mathrm{d}u] + u\int\frac{\Phi(u)}{u^4}\,\mathrm{d}u\right\}\Big/2A^2\frac{\mathrm{d}u}{\mathrm{d}\phi} \quad (31)$$

This is a second order differential equation for the orbit. Here it is interesting to recall that corresponding orbital equation in Newtonian theory is

$$\frac{\mathrm{d}u^2}{\mathrm{d}\phi^2} + u - \frac{GM}{A^2} = 0 \quad (32)$$

and the one in case of Schwartzschild metric (Einstein theory) is

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u - \frac{GM}{A^2} - 3GMu^2 = 0 \quad (33)$$

Comparing equations (30) and (33) we get an additional term in the orbital motion of the planet in this theory of gravity. This is absent in Einstein's theory. However, it is very small in magnitude.

CONCLUDING REMARKS

Assuming the Lagrangian approach is the correct way to treat f(R) theories, we have investigated the gravitational field surrounding a spherically symmetric mass distribution and the motion of a particle in this gravitational field.

In the former case the appearance of $b_0r^2/3$ in space-time metric does not allow it to be asymptotically flat when r approaches infinity. Therefore, it behaves like a contribution that comes from a cosmological constant. This is as if Einstein theory is considered with cosmological constant, that is, $R_u^v - (1/2)\delta_u^v R = -\Lambda\delta_u^v$, where Λ is the cosmological constant. This contribution is small. If $1/\sqrt{b_0} \gg r \gg GM$, the metric (19) is nearly flat. The effect of mass term M dominates for the values of r below this range and the effect of this term, $b_0r^2/3$ dominates for the values of r above this range. However, in this situation, the Newtonian potential gets modified to $\phi = -GM/r + b_0r^2/6$. The second term here appears due to the correction in the Hilbert Lagrangian, R.

It is interesting to look at the scalar curvature in an f(R) theory of gravity. For instance, equation (4) in vacuum and for $n = 2$ gives trace $\Box R - R/6a_2 = 0$. This is a wave equation and is comparable with massless scalar field equation $\Box\phi + R\phi/6 = 0$. $\Box R$ is non-vanishing for all values of $n \geq 2$. This means that scalar curvature is of wave nature in f(R) theories of gravitation.

In case f(R) is zero or constant, the metric (15) will correspond to the Schwarzschild solution of Einstein theory with or without cosmological constant.

Now we consider the motion part. The differential equation for the orbit in Einstein theory differs from the corresponding orbital equation of Newtonian theory by the term $3GMu^2$ and that of this f(R) theory differs from the corresponding orbital equation of Einstein theory by the term $\Phi(u)$. Thus the equation in this theory differs from the Newtonian theory by the terms $3GMu^2 + \Phi(u)$. The relativistic correction to planetary motion is extremely small. This can be seen by comparing second and fourth terms in equation (30). These terms differ by an order of GMu or GM/rc^2 in c.g.s. units. For Mercury $GM/rc^2 \approx 3 \times 10^{-8}$ because $M = M_\oplus = 2 \times 10^{33}$ gm and $r = 5.5 \times 10^{12}$ cm.

Since it is having the effect similar to that of a cosmological constant the change in constant GM/A^2 in equation (30) is not producing any interesting observable effect in planetary motion. However $3GMu^2$ is small compared to other terms, it is sufficient to use the method of successive approximation. We consider the solution of Newtonian equation (32) as

$$u = \frac{GM}{A^2}\{1 + \epsilon\cos(\phi - \phi_0)\} \qquad (34)$$

where ϵ and ϕ are constants. Equation (34) represents an ellipse with ϵ as eccentricity and a perihelion located at $\phi = \phi_0$. Replacing small terms $3GMu^2$ by its Newtonian approximation (34) we obtain:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u - \frac{GM}{A^2} - \frac{3(GM)^3}{A^4} - \frac{6\epsilon(GM)^3}{A^4}\cos(\phi - \phi_0) - \frac{3(GM)^3}{A^4}\epsilon^2\cos^2(\phi - \phi_0) - \bar{\Phi} = 0 \qquad (35)$$

where

$$\bar{\Phi} = \Phi\frac{GM}{A^2}[1 + \epsilon\cos(\phi - \phi_0)] \qquad (36)$$

For a nearly circular orbit, ϵ is small. We neglect the term proportional to ϵ^2. The term $3(GM)^3/A^4$ can also be neglected as it is proportional to or equivalent to the change in the constant GM/A^2 and produces no observable effects. Also $\bar{\Phi}$ can be ignored because:

$$\bar{\Phi} \approx \Phi\frac{GM}{A^2} + \phi'\frac{GM}{A^2}\left[-\frac{\epsilon GM}{A^2}\cos(\phi - \phi_0) + \frac{GM}{A^2}\right] \qquad (37)$$

The first term in (37) corresponds to the changes in constant GM/A^2 due to modified theory other than that of Einstein and is of little significance in observation. The second term in (37) vanishes at the perihelion. Therefore, the contributions from $\bar{\Phi}$ can be ignored. Thus, we have

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{A^2} - \frac{6\epsilon(GM)^3}{A^4}\cos(\phi - \phi_0) = 0 \qquad (38)$$

of which the solution is

$$u = \frac{GM}{A^2}[1 + \epsilon\cos(\phi - \phi_0)] + \frac{3\epsilon(GM)^3}{A^4}\phi\sin(\phi - \phi_0) \qquad (39)$$

or it can be written as

$$u = \frac{GM}{A^2}\left[1 + \epsilon\cos\{\phi - \phi_0 - 3\frac{G^2M^2}{A^2}\phi\}\right] \qquad (40)$$

Equation (40) represents a precessing elliptical orbit. If ϕ changes by

$$\Delta\phi = 2\pi\left[1 - \frac{3G^2M^2}{A^2}\right]^{-1} \approx 2\pi\left[1 + \frac{3G^2M^2}{A^2}\right] \qquad (41)$$

the arguments of cosine changes by 2π. This shows that the angular distance between one perihelion and the next is larger than 2π by $6\pi G^2M^2/A^2$. This quantity gives the angular precession of the perihelion per revolution showing that the perihelion advances $\Delta\phi > 2\pi$ in the direction of motion. There is no further need to proceed for approximation.

For a nearly circular orbit, equation (34) gives $GM/A^2 = 1/r$ where r is the radius of the orbit. The angular advance of the perihelion per revolution in c.g.s. unit is $6\pi GM/rc^2$ (42)

Thus, in this f(R) theory of gravity observable effects are similar to that of Einstein theory as the term $\Phi(u)$ has nothing to contribute even in successive approximations. Therefore, the precession of the perihelion is consistence with observation. One of the possible reasons for this can be seen in the fact that in this choice of f(R), the resulting field equation (4) is based only on the scalar curvature and is not associated with any other field like scalar field or meson field.

It is interesting to note that the precession of the orbit can be quite large in case of close binary star systems. For a system consisting of two white dwarfs or two neutron stars of mass $1M_{\oplus}$ separated by a distance of 10^{11} cm, equation (42) gives a periastron advance of 3×10^{-5} radians per revolution which means $\approx$ 2°per year.

We have considered a choice of f(R) assuming that Einstein general relativity is the correct theory of gravity. On the contrary, if f(R) theories are indeed able to explain the accelerated expansion the right choice for the function f(R) and how the variation has to be performed (higher order metric or Palatini approach) should be investigated. One can expect that the functional expression of f(R) is not changing during evolution of the universe, even if R evolves with cosmic time. If this is the case then f(R) theory should reproduce the phenomenology we observe to day but also give rise to an inflationary period in the early universe. Then, the logarithmic Lagrangian can be ignored because it does not predict any inflationary period, whereas the choice $f(R) = \beta R^n$ is able to explain inflation provided one sets $n = 2$.

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